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Exact Seiberg-Witten Map, Induced Gravity and Topological Invariants in Noncommutative Field Theories

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ABSTRACT

We revisit the exact Seiberg-Witten (SW) map on Dirac-Born-Infeld actions, making a connection with the deformation quantization scheme. The picture on field dependent induced gravity from noncommutativity becomes more transparent in the context of deformation quantization. We also find an exact SW map for an adjoint scalar field, consistent with that deduced from RR couplings of unstable non-BPS D-branes. The dual description via the exact SW map can again be interpreted as the ordinary field theory coupling to gravity induced by gauge fields. Using the exact SW maps, we further discuss several aspects of topological invariants in noncommutative (NC) gauge theory. Especially, it is shown that the K-theory class on NC instantons is mapped to the usual second Chern class via exact SW map and it leads to an exact SW map between commutative and NC Chern-Simons terms.

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1 Introduction

A noncommutative (NC) space is obtained by quantizing a given space with its symplectic structure, treating it as a phase space. Also field theories can be formulated on a NC space. NC field theory means that fields are defined as functions over the NC space. At the algebraic level, the fields become operators acting on a Hilbert space as a representation space of the NC space. Since the NC space resembles a quantized phase space, the idea of localization in ordinary field theory is lost. The notion of a point is replaced by that of a state in the representation space. Thus it may help understanding non-locality at short distances in quantum gravity.

Recently it has been known [1, 2] that NC field theories can arise naturally as a decoupled limit of open string dynamics on D-branes in the background of a Neveu-Schwarz B field. The open string effective action on a D-brane is given by the Dirac-Born-Infeld (DBI) action in the limit of slowly varying fields [3]. Seiberg and Witten, however, showed [4] that an explicit form of the effective action depends on the regularization scheme of two dimensional field theory defined by the worldsheet action. That is, depending on the regularization scheme or path integral prescription for the open string ending on a D-brane, one can have two descriptions: commutative and NC descriptions. Since these two descriptions arise from the same open string theory depending on different regularizations and the physics should not depend on the regularization scheme, Seiberg and Witten argued [4] that the two descriptions should be equivalent and thus there must be a spacetime field redefinition between ordinary and NC gauge fields, so called Seiberg-Witten (SW) map.

In this sense NC gauge theories have a dual description through the SW map in terms of ordinary gauge theory on commutative spacetime. To understand the dual description exactly, it is important to know the exact SW map between the gauge fields. In a recent work [5], it was pointed out that there is an extremely simple way to find the exact SW map using the change of variables between the open and closed string parameters. The resulting exact SW map revealed a remarkable picture that the NC Maxwell action can be regarded as the ordinary Maxwell action coupling to a metric deformed by gauge fields, which genuinely realizes an interesting idea by Rivelles [6].

Why NC gauge fields play a role of gravity may be understood by noting [7, 8] that translations in the NC directions are equivalent to a gauge transformation up to global symmetry transformations. Based on this property the authors in [7] assert that NC gauge theories are toy models of general relativity. We quote a paragraph in [7]:

> What is unusual about noncommutative gauge theories is that *translations in the noncommutative directions are equivalent to a combination of a gauge transformation and a constant shift of the gauge field.* This explains why in noncommutative gauge theories there do not exist local gauge invariant observables, since by a gauge transformation we can effect a spatial translation ! This is analogous to the situation

in general relativity, where translations are also equivalent to gauge transformations (general coordinate transformations) and one cannot construct local gauge invariant observables. The fact that spatial translations are equivalent to gauge transformations (up to global symmetry transformations) is one of the most interesting features of noncommutative gauge theories. These theories are thus toy models of general relativity - the only other theory that shares this property.

If one employs commutative description via SW map, however, the connection between translations and gauge transformations is lost. A global translation on commutative fields can no longer be rewritten as a gauge transformation. So one may wonder how the property of NC field theories show up in the commutative description via SW map. Now the aspect concerning gravity directly emerges as an effective metric induced by gauge fields when the commutative description is employed. Indeed this was the motivation in [6] to explore the connection between NC field theories and gravity.

This paper is organized as follows. In Sec. 2, we briefly summarize the exact SW maps on DBI actions obtained in [5] to make a connection with later sections. In Sec. 3, we adopt the deformation quantization scheme a la Kontsevich [9] to show that the results in [5] can be reproduced in this approach too. The picture on the induced gravity from noncommutativity becomes more transparent in the context of deformation quantization. In Sec. 4, we find an exact SW map for a scalar field in the adjoint representation of gauge group and show that it is consistent with that deduced from RR couplings of unstable non-BPS D-branes [10]. The dual description via the exact SW map can again be interpreted as the ordinary field theory coupling to dilaton gravity induced by gauge fields. In Sec. 5, we discuss several aspects of topological invariants in NC gauge theory using the exact SW maps. Especially, it is shown that the K-theory class on NC instantons is mapped to the usual second Chern class via exact SW map and it leads to an exact SW map between commutative and NC Chern-Simons terms, which was proved earlier in [11]. In Sec. 6, we briefly summarize our results obtained and discuss some related open issues.

2 Exact Seiberg-Witten Map and Induced Gravity from Noncommutativity

In this section, we recapitulate the exact SW maps on DBI actions obtained in [5] to make a connection with later sections. The worldsheet action governing the open string dynamics attached on Dp-branes in flat spacetime, with metric $g_{\mu\nu}$, in the presence of a constant Neveu-Schwarz B-field is given by

$$S = \frac{1}{2\kappa}\int_\Sigma d^2\sigma g_{\mu\nu}\partial_a x^\mu \partial^a x^\nu + i\int_{\partial\Sigma} d\tau \Big(\frac{1}{2}B_{\mu\nu}x^\nu - A_\mu(x)\Big)\partial_\tau x^\mu, \tag{2.1}$$

where the string worldhseet Σ is the upper half plane parameterized by $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma \leq \infty$ and $\partial\Sigma$ is its boundary. We define the inverse string tension as

$$\kappa \equiv 2\pi\alpha', \tag{2.2}$$

which is a useful expansion parameter in a low energy effective action of D-branes. The propagator evaluated at boundary points [4] is

$$\langle x^\mu(\tau)x^\nu(\tau')\rangle = -\frac{\kappa}{2\pi}\Big(\frac{1}{G}\Big)^{\mu\nu}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau') \tag{2.3}$$

where $\epsilon(\tau)$ is the step function. Here

$$\Big(\frac{1}{G}\Big)^{\mu\nu} = \Big(\frac{1}{g+\kappa B}g\frac{1}{g-\kappa B}\Big)^{\mu\nu}, \tag{2.4}$$

$$G_{\mu\nu} = g_{\mu\nu} - \kappa^2(Bg^{-1}B)_{\mu\nu}, \tag{2.5}$$

$$\theta^{\mu\nu} = -\kappa^2\Big(\frac{1}{g+\kappa B}B\frac{1}{g-\kappa B}\Big)^{\mu\nu}. \tag{2.6}$$

In what follows, we will often use the matrix notation:

$$AB = A_{\mu\alpha}B^{\alpha\mu}, \quad (AB)_{\mu\nu} = A_{\mu\alpha}B^{\alpha}{}_{\nu}, \quad \text{etc.} \tag{2.7}$$

¿From Eqs. (2.4) and (2.6), we have the following relation

$$\frac{1}{G} + \frac{\theta}{\kappa} = \frac{1}{g+\kappa B}. \tag{2.8}$$

The object $G_{\mu\nu}$ has a simple interpretation as the effective metric seen by the open strings while $g_{\mu\nu}$ is the closed string metric. Furthermore the coefficient $\theta^{\mu\nu}$ has a simple interpretation as

$$[x^\mu(\tau), x^\nu(\tau)] = i\theta^{\mu\nu}. \tag{2.9}$$

That is, x^μ are coordinates on a NC space with noncommutativity parameter θ [1].

For a slowly varying approximation of neglecting derivative terms, i.e., $\sqrt{\kappa}|\frac{\partial F}{F}| \ll 1$, the spacetime low energy effective action on a single Dp-brane is given by the DBI action [3]

$$S(g_s, g, A, B) = \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}}\int d^{p+1}x\sqrt{-\det(g+\kappa(B+F))}, \tag{2.10}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.11}$$

Note that the effective action is expressed in terms of closed string variables $g_{\mu\nu}, B_{\mu\nu}$ and g_s. Seiberg and Witten, however, showed [4] that an explicit form of the effective action depends on the regularization scheme of two dimensional field theory defined by the worldsheet action (2.1), which is related to field redefinition in spacetime.

As was explained in [4], there is a general description with an arbitrary θ associated with a suitable regularization that interpolates between Pauli-Villars and point-splitting. This freedom is basically coming from the fact that the sigma model (2.1) has a symmetry $A \to A + \Lambda$, $B \to B - d\Lambda$, for any one-form Λ and thus the open string theory depends only on the gauge invariant combination $\mathcal{F} = B + F$. Given such a symmetry, there is a freedom of shift in B keeping fixed $\mathcal{F}$. By taking the background to be B or B', we get a NC description with appropriate θ or θ', and different F's. The freedom in the description is parameterized by a two-form Φ. In this case the change of variables found by Seiberg and Witten [4] is given by

$$\frac{1}{G + \kappa\Phi} + \frac{\theta}{\kappa} = \frac{1}{g + \kappa B}, \tag{2.12}$$

$$G_s = g_s \sqrt{\frac{\det(G + \kappa\Phi)}{\det(g + \kappa B)}}. \tag{2.13}$$

The effective action in these variables is given by

$$\widehat{S}_\Phi(G_s, G, \widehat{A}, \theta) = \frac{2\pi}{G_s (2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{-\det(G + \kappa(\widehat{F} + \Phi))}. \tag{2.14}$$

The action depends on the open string variables $G_{\mu\nu}, \theta_{\mu\nu}$ and G_s, where the θ-dependence is entirely in the $\star$ product in the field strength $\widehat{F}$:

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i\widehat{A}_\mu \star \widehat{A}_\nu + i\widehat{A}_\nu \star \widehat{A}_\mu. \tag{2.15}$$

For every background characterized by $B, g_{\mu\nu}$ and g_s, we thus have a continuum of descriptions labelled by a choice of Φ. Indeed, for $\Phi = B$ where $G = g$, $G_s = g_s$ and $\theta = 0$, $\widehat{S}_\Phi$ recovers the commutative description (2.10) while $\Phi = 0$ leads to the familiar NC description. Seiberg and Witten [4] proved that DBI actions are independent of the choice Φ, namely,

$$\widehat{S}_\Phi(G_s, G, \widehat{A}, \theta) = S(g_s, g, A, B) + \mathcal{O}(\partial F). \tag{2.16}$$

It was shown in [5] that the dual description through the exact SW map is simply given by the identity (2.16) using the change of variables between open and closed string parameters, (2.12) and (2.13). More precisely, the dual description of the NC DBI action (2.14) via the exact SW map is given by the ordinary one (2.10) expressed in terms of open string variables

$$\begin{aligned} &\int d^{p+1}x \sqrt{-\det(G + \kappa(\widehat{F} + \Phi))} \\ &\qquad = \int d^{p+1}x \sqrt{\det(1 + F\theta)} \sqrt{-\det(G + \kappa(\Phi + \mathbf{F}))} + \mathcal{O}(\partial F), \end{aligned} \tag{2.17}$$

where

$$\mathbf{F} = \frac{1}{1 + F\theta} F. \tag{2.18}$$

Note that the action (2.17) is exactly the same as the DBI action obtained using the ζ-function regularization scheme by Andreev and Dorn [12] (their Eq. (2.24)).

In the zero slope limit $\kappa \to 0$, Eq. (2.17) for $\Phi = 0$ and $p = 3$ defines an exact nonlinear action of the SW deformed electrodynamics:

$$-\frac{1}{4g_{YM}^2}\int d^4x \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} = -\frac{1}{4g_{YM}^2}\int d^4x \sqrt{-\mathrm{detg}}\ \mathrm{g}^{\mu\alpha}\mathrm{g}^{\beta\nu}F_{\mu\nu}F_{\alpha\beta}, \tag{2.19}$$

where we introduced an effective non-symmetric "metric" induced by the dynamical gauge fields such that

$$\mathrm{g}_{\mu\nu} = \eta_{\mu\nu} + (F\theta)_{\mu\nu}, \qquad \mathrm{g}^{\mu\nu} = \Big(\frac{1}{\eta + F\theta}\Big)^{\mu\nu}. \tag{2.20}$$

The NC Maxwell action after the SW map looks like the ordinary Maxwell theory coupled to the "induced metric" $\mathrm{g}_{\mu\nu}$. It should be remarked that the gravitational field in the action (2.19) cannot be interpreted just as a fixed background since it depends on the dynamical gauge field. The identity (2.19) is very remarkable in the sense that the NC Maxwell action after the exact SW map can be regarded as an ordinary field theory coupling to a field dependent gravitational background [6].

A simple yet nontrivial application of the mapping (2.19) is in the context of conformal anomalies. The planar part of the conformal anomaly in NC gauge theory is known [13] to be proportional to the left hand side of Eq. (2.19). Then using the map it is feasible to express the result in terms of commutative variables. In this way the conformal anomalies in the NC and commutative descriptions get related. We might recall that current (divergence) anomalies in NC and commutative theories are also related by appropriate SW maps [14].

Another interesting case arises from the choice $\Phi_{\mu\nu} = -B_{\mu\nu}$, which naturally appears in the matrix model [4, 15]. In this case, with the Euclidean signature,

$$\theta = \frac{1}{B}, \qquad G = -\kappa^2 B\frac{1}{g}B, \qquad G_s = g_s\sqrt{\det(-\kappa B g^{-1})} \tag{2.21}$$

and

$$(\widehat{F} + \Phi)_{\mu\nu} = iB_{\mu\lambda}[X^\lambda, X^\sigma]_\star B_{\sigma\nu}, \tag{2.22}$$

where

$$X^\mu = x^\mu + \theta^{\mu\nu}\widehat{A}_\nu. \tag{2.23}$$

The DBI action related to the matrix model has more natural description, so called, background independent formulation, in terms of closed string variables [15]. The NC DBI action (2.14) can be expressed instead in terms of closed string variables using the relation (2.21) and then the equivalence (2.16) defines the exact inverse SW map

$$\begin{aligned}&\frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}}\int d^{p+1}x\sqrt{\det(g+\kappa\mathcal{F})}\\ &= \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}}\int d^{p+1}x\sqrt{\det(1-\theta\widehat{F})}\sqrt{\det(g+\kappa(B+\widehat{\mathbf{F}}))},\end{aligned} \tag{2.24}$$

where

$$\widehat{\mathbf{F}} = \widehat{F}\frac{1}{1-\theta\widehat{F}}. \tag{2.25}$$

Our result (2.24) is consistent with the exact SW map obtained by completely independent way in [16, 17, 10, 18] as shown in next section.

Note that

$$B + \widehat{\mathbf{F}} = B\frac{1}{1-\theta\widehat{F}}. \tag{2.26}$$

In the zero slope limit, $\kappa \to 0$, now keeping fixed $g_{\mu\nu}$ and g_{YM}^2, we obtain an intriguing identity

$$\frac{1}{4g_{YM}^2}\int d^{p+1}x \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = \frac{1}{4g_{YM}^2}\int d^{p+1}x\sqrt{\det\widehat{\mathrm{g}}}\; \widehat{\mathrm{g}}^{\mu\alpha}\widehat{\mathrm{g}}^{\beta\nu}B_{\mu\nu}B_{\alpha\beta}, \tag{2.27}$$

where

$$\widehat{\mathrm{g}}_{\mu\nu} = \delta_{\mu\nu} - (\theta\widehat{F})_{\mu\nu}, \qquad \widehat{\mathrm{g}}^{\mu\nu} = \Big(\frac{1}{1-\theta\widehat{F}}\Big)^{\mu\nu}. \tag{2.28}$$

The identity (2.27) definitely shows that fluctuations F with respect to the background B induce fluctuations of (NC) geometry from matrix model side.

3 Geometric Construction of Exact Seiberg-Witten Map

In this section we will demonstrate that the results in the previous section can be reproduced in the context of deformation quantization [9]. First let us start with a brief recapitulation of the results in [19, 20]. Consider two symplectic forms

$$\omega_{\mu\nu} = (\theta^{-1})_{\mu\nu} + F_{\mu\nu}, \qquad B_{\mu\nu} = (\theta^{-1})_{\mu\nu} \tag{3.1}$$

where $\theta^{\mu\nu}$ is a constant anti-symmetric tensor and $F = dA$ is the field strength of Abelian gauge field A_μ. We shall assume that both ω and B are non-degenerate. We can associate star products $\star_\omega$ and $\star_B$ with ω and B, respectively, in the context of the deformation quantization a la Kontsevich [9]. The SW map is expressed in terms of a transformation which relates the star product associated with B to the one associated with ω.

Since B and ω differ by an exact form, it is possible to find a coordinate transformation ρ which maps ω to B, i.e., $\rho : x \to y = y(x)$ so that

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\omega_{\alpha\beta}(y) = B_{\mu\nu}. \tag{3.2}$$

Thus the symplectic structures defined by ω and B belong to the same equivalence class and the two star products $\star_\omega$ and $\star_B$ must be equivalent. (Eq. (3.2) is essentially the statement of Darboux theorem on symplectic manifolds. We refer Sects. 3.2 and 3.3 in [21] for a proof of this theorem and related symplectic geometry.) In other words, we can eliminate any fluctuation

of the electromagnetic field strength by a simple coordinate redefinition. More explicitly, there exists a map $\mathcal{D}$ acting on the space of functions which satisfies

$$\mathcal{D}(f \star_\omega g) = \mathcal{D}f \star_B \mathcal{D}g. \tag{3.3}$$

In particular the NC gauge field is defined by

$$X^\mu(x) = \mathcal{D}y^\mu \equiv x^\mu + \theta^{\mu\nu}\widehat{A}_\nu. \tag{3.4}$$

This change of coordinates is not unique, but is defined up to diffeomorphisms, the canonical transformations or symplectomorphisms, which preserve θ. The group of such diffeomorphisms is non-Abelian and is generated by Hamiltonian vector fields of the form $\delta x^\mu = \{x^\mu, S\}_\theta$ for some generating function S. These diffeomorphisms replace the ordinary Abelian gauge invariance of the original theory. That is, the NC gauge group is the set of diffeomorphisms which leaves the two-form B invariant. Within the framework of Kontsevich's deformation quantization, the equivalence classes of Poisson manifolds can thus be naturally identified with the sets of gauge equivalence classes of star products on a Poisson manifold M [20], which leads to the SW transformations. An important lesson from the above arguments is that, in the new coordinate system y^μ, the dynamics is not described by the $U(1)$ gauge potential $A_\mu(x)$, but is described by the embedding functions $x^\mu(y)$ which are now the dynamical fields [19]. In this way any fluctuation of the field strength can be eliminated in favor of fluctuations of the induced metric. This explains why the gauge fields play a role of gravity.

In Eq. (3.3), setting $f(y) = y^\mu$ and $g(y) = y^\nu$ and supposing that the product between functions in the left and right hand sides is defined by star products $\star_\omega$ and $\star_B$, respectively, we get

$$(\omega^{-1})^{\mu\nu}(X(x)) = (\theta - \theta\widehat{F}\theta)^{\mu\nu}(x) \tag{3.5}$$

where the NC field strength $\widehat{F}_{\mu\nu}$ is given by Eq. (2.15). Eqs. (3.1) and (3.5) then lead to [16]

$$F_{\mu\nu}(X) = \Big(\widehat{F}\frac{1}{1-\theta\widehat{F}}\Big)_{\mu\nu}(x) \tag{3.6}$$

or its inverse

$$\widehat{F}_{\mu\nu}(x) = \Big(\frac{1}{1+F\theta}F\Big)_{\mu\nu}(X). \tag{3.7}$$

Note that

$$\widehat{F}\frac{1}{1-\theta\widehat{F}} = \frac{1}{1-\widehat{F}\theta}\Big(\widehat{F} - \widehat{F}\theta\widehat{F}\Big)\frac{1}{1-\theta\widehat{F}} = \frac{1}{1-\widehat{F}\theta}\widehat{F} \tag{3.8}$$

and similarly for Eq. (3.7). Antisymmetricity of $F_{\mu\nu}$ and $\widehat{F}_{\mu\nu}$ is guaranteed due to this property.

Since

$$F_{\mu\nu}(k) = \int d^{p+1}X F_{\mu\nu}(X)e^{ik\cdot X}, \tag{3.9}$$

we obtain

$$F_{\mu\nu}(k) = \int d^{p+1}x\sqrt{\det(1-\theta\widehat{F})}\Big(\widehat{F}\frac{1}{1-\theta\widehat{F}}\Big)_{\mu\nu}(x)e^{ik\cdot X}, \tag{3.10}$$

where we used the formula

$$d^{p+1}X = d^{p+1}x\sqrt{\det(1-\theta\widehat{F})} \tag{3.11}$$

which can be derived from Eqs. (3.2) and (3.5). Note also that it follows from Eq. (3.5) that

$$\sqrt{\det(1-\theta\widehat{F}(x))}\sqrt{\det(1+F(X)\theta)} = 1. \tag{3.12}$$

Using the formula [22]

$$\begin{aligned} e^{ik\cdot X} &= P_\star \exp\Big(i\int_0^1 d\tau \partial_\tau \xi^\mu(\tau)\widehat{A}_\mu(x+\xi(\tau))\Big) \star e^{ik\cdot x}, \\ &= W(x, C_k) \star e^{ik\cdot x}, \end{aligned} \tag{3.13}$$

where $P_\star$ denotes path ordering with respect to the $\star$-product and $W(x, C_k)$ is a straight open Wilson line with path C_k parameterized by

$$\xi^\mu(\tau) = \theta^{\mu\nu}k_\nu \tau, \tag{3.14}$$

we can get the exact (inverse) SW map for the field strength [16, 17, 10, 18]

$$F_{\mu\nu}(k) = \int d^{p+1}x\sqrt{\det(1-\theta\widehat{F})}\Big(\widehat{F}\frac{1}{1-\theta\widehat{F}}\Big)_{\mu\nu}(x)W(x, C_k) \star e^{ik\cdot x}. \tag{3.15}$$

It is easy to demonstrate the equivalence of the DBI actions, Eq. (2.16), using Eqs. (2.12), (2.13), (3.6), (3.7), (3.11), and (3.12) [16, 23]:

$$\begin{aligned} &\frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}}\int d^{p+1}X\sqrt{-\det(g+\kappa(F(X)+B))} \\ &\qquad = \frac{2\pi}{G_s(2\pi\kappa)^{\frac{p+1}{2}}}\int d^{p+1}x\sqrt{-\det(G+\kappa(\widehat{F}(x)+\Phi))}. \end{aligned} \tag{3.16}$$

Here we used different coordinates, X^μ and x^μ, for commutative and NC descriptions, respectively. There is a simple reason for the use of these coordinates. A natural coordinate system in the commutative description is θ-independent, i.e. background independent, one, which is X-coordinates [15], while that in the NC one is x-coordinates due to their simple commutation relation (2.9). It is also very easy to prove the exact SW maps, Eqs. (2.17) and (2.24), using Eqs. (3.6), (3.7), (3.11), and (3.12).

We here discuss the SW map for constant field strength. Note that the arguments in this section should also hold for this case. Thus we see that the exact SW maps, Eqs. (2.17) and (2.24), must be true even for the constant field strength. If we thus take the solution (3.7) (which is exact for constant fields) plus the measure factor coming from Eq. (3.11), then a simple correspondence immediately leads to Eq. (2.19), thereby proving the identity for constant fields exactly. The measure change thus correctly accounts for those terms which would otherwise be dropped in the constant field approximation. This is in conformity with similar observations [16] needed to show the equivalence of DBI actions.

4 Exact Seiberg-Witten Map for Scalar Fields

In this section we will find an exact SW map for scalar fields in the adjoint representation of gauge group. To get it, we will apply the standard dimensional reduction scheme for the exact SW map, Eq. (3.6) or (3.7). For this purpose, we compactify one of the spatial directions and we denote its compact coordinate as $z \in \mathbf{S}^1$ and the compact gauge fields along $\mathbf{S}^1$ as $\widehat{A}_z = \widehat{\varphi}$ and $A_z = \varphi$. According to the usual dimensional reduction scheme, we set $\theta^{\mu z} = -\theta^{z\mu} = 0$ where μ spans non-compact NC directions.

Adopting the standard rule, we identify

$$\begin{aligned} \widehat{F}_{\mu z}(x) &= \partial_\mu \widehat{\varphi} - i[\widehat{A}_\mu, \widehat{\varphi}]_\star \\ &= \widehat{D}_\mu \star \widehat{\varphi}(x), \end{aligned} \tag{4.1}$$

$$F_{\mu z}(X) = \frac{\partial \varphi(X)}{\partial X^\mu}. \tag{4.2}$$

For the exact SW map (3.7), we take the ordering

$$\begin{aligned} \widehat{D}_\mu \star \widehat{\varphi}(x) &= \widehat{F}_{\mu z}(x) \\ &= \Big(\frac{1}{1+F\theta}\Big)_\mu^{\ \nu}(X) \frac{\partial \varphi(X)}{\partial X^\nu} \end{aligned} \tag{4.3}$$

since

$$\widehat{F}_{z\mu}(x) = \Big(\frac{F\theta}{1+F\theta}\Big)_\mu^{\ \nu}(X) \frac{\partial \varphi(X)}{\partial X^\nu} \tag{4.4}$$

does not produce the correct commutative limit when $\theta \to 0$. We regard Eq. (4.3) as the exact SW map for the adjoint scalar field $\widehat{\varphi}$. It is straightforward to generalize to the case being several adjoint scalar fields $\widehat{\varphi}_i$, $i = 1, \cdots, n$, by considering a similar dimensional reduction onto $\mathbf{T}^n$.

Now we will show that the SW map (4.3) obtained by the dimensional reduction scheme is consistent with that obtained by studying RR couplings of unstable non-BPS D-branes [10]. Consider the coupling of a non-BPS Dp-brane to the RR form $C^{(p)}$ in the commutative description

$$\begin{aligned} \int dT \wedge C^{(p)} &= \int d^{p+1}X \varepsilon^{\mu_1\mu_2\cdots\mu_{p+1}} \mathcal{O}_{\mu_1}(X) C^{(p)}_{\mu_2\cdots\mu_{p+1}}(X) \\ &= \int d^{p+1}k \varepsilon^{\mu_1\mu_2\cdots\mu_{p+1}} \widetilde{\mathcal{O}}_{\mu_1}(k) \widetilde{C}^{(p)}_{\mu_2\cdots\mu_{p+1}}(-k), \end{aligned} \tag{4.5}$$

where T is the tachyon field and

$$\mathcal{O}_\mu(X) = \frac{\partial T(X)}{\partial X^\mu}. \tag{4.6}$$

The same RR coupling $C^{(p)}$ of a NC non-BPS Dp-brane has the following form for each momentum mode [10]

$$\varepsilon^{\mu_1\mu_2\cdots\mu_{p+1}} \widetilde{C}^{(p)}_{\mu_2\cdots\mu_{p+1}}(-k) \int \frac{d^{p+1}x}{(2\pi)^{p+1}} L_\star\Big[\sqrt{\det(1-\theta\widehat{F})}\widehat{\mathcal{O}}_{\mu_1}(x) W(x, C_k)\Big] \star e^{ik\cdot x} \tag{4.7}$$

where

$$\widehat{\mathcal{O}}_\mu(x) = \Big(\frac{1}{1-\widehat{F}\theta}\Big)_\mu^{\ \nu}(x)(\widehat{D}_\nu \star \widehat{T})(x). \tag{4.8}$$

From the equivalence of commutative and NC couplings to the RR form $C^{(p)}$ of a non-BPS Dp-brane, we find that

$$\widetilde{\mathcal{O}}_\mu(k) = \int \frac{d^{p+1}x}{(2\pi)^{p+1}} L_\star\Big[\sqrt{\det(1-\theta\widehat{F})}\widehat{\mathcal{O}}_\mu(x)W(x,C_k)\Big] \star e^{ik\cdot x}. \tag{4.9}$$

In the DBI approximation, we get

$$\frac{\partial T(X)}{\partial X^\mu} = \Big(\frac{1}{1-\widehat{F}\theta}\Big)_\mu^{\ \nu}(x)(\widehat{D}_\nu \star \widehat{T})(x) \tag{4.10}$$

where we used Eqs. (3.11) and (3.13). This SW map can simply be obtained from Eq. (3.6) by the dimensional reduction (4.1) and (4.2), thus proving the consistency of our scheme.

For a real scalar field in the adjoint representation of $U(1)$, the flat spacetime action for the NC scalar field is

$$\widehat{S}_{\widehat{\varphi}} = \frac{1}{2}\int d^4x \widehat{D}^\mu\widehat{\varphi} \star \widehat{D}_\mu\widehat{\varphi}. \tag{4.11}$$

The action is invariant under the gauge transformation

$$\widehat{\delta}_{\widehat{\lambda}}\widehat{A}_\mu = \widehat{D}_\mu \star \widehat{\lambda}, \qquad \widehat{\delta}_{\widehat{\lambda}}\widehat{\varphi} = -i[\widehat{\varphi},\widehat{\lambda}]_\star. \tag{4.12}$$

We apply the exact SW map (4.3) to the action (4.11) and the result is

$$\frac{1}{2}\int d^4x \widehat{D}^\mu\widehat{\varphi} \star \widehat{D}_\mu\widehat{\varphi} = \frac{1}{2}\int d^4x \sqrt{\det(1+F\theta)}\Big(\frac{1}{1+F\theta}\frac{1}{1+\theta F}\Big)^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi. \tag{4.13}$$

Here we are using the same symbol x to denote both the commutative (the right hand side) and the NC (the left hand side) coordinates. We will often use the symbol x for both descriptions when the distinction is not necessary. It can be easily checked that the leading order in θ in the right hand side of Eq. (4.13) exactly coincides with Eq. (7) in [6].

The final form (4.13) after the exact SW map can be recast to the form coupled to a gravitational background with a specific dilaton coupling:

$$S_\varphi = \frac{1}{2}\int d^4x e^{-\phi}\sqrt{\det \mathrm{g}}\, \mathrm{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \tag{4.14}$$

where we introduced an induced symmetric metric

$$\mathrm{g}_{\mu\nu} = \Big((1+\theta F)(1+F\theta)\Big)_{\mu\nu}, \qquad \mathrm{g}^{\mu\nu} = \Big(\frac{1}{1+F\theta}\frac{1}{1+\theta F}\Big)^{\mu\nu} \tag{4.15}$$

and a dilaton given by

$$\phi = \frac{1}{4}\mathrm{Tr}\,\ln \mathrm{g}. \tag{4.16}$$

Our metric coupling (4.14) is different from Eq. (8) in [6]. (It is not possible to have the symmetric traceless metric such as Eq. (9) in [6] beyond the leading order in θ.) The metric in Einstein relativity is a property of spacetime itself rather than a field over spacetime and thus all non-gravitational fields should couple in the same manner to a single gravitational field, sometimes called "universal coupling". We here see that the metric coupling induced by noncommutativity is not universal, i.e., species dependent. However, this is somewhat expected since, in NC field theory context, there does not exist a principle to guarantee the universal coupling such as the Equivalence principle in Einstein relativity. Although we could not yet find the exact SW map for a scalar field in the fundamental representation of gauge group, we think that the same thing also happens for that case.

5 Topological Invariants and Exact Seiberg-Witten Map

The coupling of D-branes to RR potentials [16, 17, 10, 18] is given by

$$S_{WZ} = \int d^{p+1}k Q(k) D(-k) \tag{5.1}$$

where $D = Ce^{\frac{\kappa}{2\pi}B}$ and

$$Q(k) = \int d^{p+1}x L_\star \Big[\sqrt{\det(1-\theta\widehat{F})} e^{\frac{\kappa}{2\pi}\widehat{F}\frac{1}{1-\theta\widehat{F}}} W(x, C_k)\Big] \star e^{ik\cdot x}. \tag{5.2}$$

$Q(k=0)$ defines the charges of lower dimensional branes which can be identified with a K-theory class $\mu(E)$ of a projective module E as an element of integral even cohomology [18]. One can immediately see that $Q(k=0)$ in terms of commutative coordinates X maps to the (integrated) Chern character $\mathrm{ch}(E)$ by Eqs. (3.6) and (3.11), i.e.,

$$\begin{aligned} \mu(E) &= \int d^{p+1}x \sqrt{\det(1-\theta\widehat{F})} e^{\frac{\kappa}{2\pi}\widehat{F}\frac{1}{1-\theta\widehat{F}}} \\ &= \int d^{p+1}X e^{\frac{\kappa}{2\pi}F(X)} \\ &= \int d^{p+1}X \mathrm{ch}(E). \end{aligned} \tag{5.3}$$

The identity (5.3) directly proves that the K-theory class $\mu(E)$ of a projective module E takes values in an integral even cohomology class. We should emphasize that the whole argument in this section is equally valid even for non-Abelian case although we present only Abelian case for simplicity.

The identity (5.3) for four dimensions is related to the topological charge of instantons

$$\frac{1}{64\pi^2}\int d^4x \sqrt{\det(1-\theta\widehat{F})}\Big(\widehat{F}\frac{1}{1-\theta\widehat{F}}\Big) \wedge \Big(\widehat{F}\frac{1}{1-\theta\widehat{F}}\Big) = \frac{1}{64\pi^2}\int d^4X F\wedge F \tag{5.4}$$

where we used the (star) wedge notation

$$\widehat{F} \wedge \cdots \wedge \widehat{F} = \varepsilon^{\mu\nu\cdots\lambda\rho} \widehat{F}_{\mu\nu} \star \cdots \star \widehat{F}_{\lambda\rho}. \tag{5.5}$$

Related to the instanton number in NC gauge theory, the quantity one usually calculates has the form instead

$$\frac{1}{64\pi^2} \int d^4x \widehat{F} \wedge \widehat{F} \tag{5.6}$$

and it has been known [24] that it is also integer valued. So it is natural to expect that

$$\int d^4x \sqrt{\det \widehat{\mathrm{g}}} \Big(\widehat{F} \widehat{\mathrm{g}}^{-1} \Big) \wedge \Big(\widehat{F} \widehat{\mathrm{g}}^{-1} \Big) = \int d^4x \widehat{F} \wedge \widehat{F} \tag{5.7}$$

with the induced metric (2.28). We will prove the identity (5.7) on a more general ground. First note that the θ dependence in $\mu(E)$ comes from the explicit dependence on θ as well as the implicit one through the definition of $\widehat{F}$ and the $\star$-product between them. However, since we are working in the limit of slowly varying approximation which is constantly assumed in the derivation of DBI actions in string theory, we can ignore all derivatives of $\widehat{F}$ and regard the products in the expansion of $\mu(E)$ as ordinary products, i.e. the implicit θ dependence is only in the definition of $\widehat{F}$ [4]. To correctly incorporate the derivatives of $\widehat{F}$, it is necessary to systematically include higher order α' corrections to both descriptions. Indeed it was known [25] that there is such an α' correction in (5.4).

Now we will show that the explicit θ dependence in $\mu(E)$ actually vanishes in the approximation of neglecting derivatives of $\widehat{F}$. Taking the derivatives with respect to the explicit dependence, we get

$$\delta_\theta \mu(E) = \int d^{p+1}x \sqrt{\det(1 - \theta \widehat{F})} \Big(-\mathrm{Tr}\, (\delta\theta \widehat{\mathbf{F}}) e^{\frac{\kappa}{2\pi} \widehat{\mathbf{F}}} + \frac{\kappa}{\pi} (\widehat{\mathbf{F}} \delta\theta \widehat{\mathbf{F}}) \wedge e^{\frac{\kappa}{2\pi} \widehat{\mathbf{F}}} \Big) = 0 \tag{5.8}$$

where we used the identity [26],

$$\theta^{\alpha\beta} F_{\alpha\beta} (F \wedge \cdots \wedge F)_{\mathrm{n-fold}} = -2n (F\theta F) \wedge (F \wedge \cdots \wedge F), \tag{5.9}$$

which is valid for any antisymmetric tensors. This means that there is no explicit dependence on θ, so that $\mu(E)$ is more simplified by setting $\theta = 0$ whenever it occurs explicitly. That is, as was first shown in [26],

$$\begin{aligned} \mu(E) &= \int d^{p+1}x \sqrt{\det(1 - \theta \widehat{F})} e^{\frac{\kappa}{2\pi} \widehat{\mathbf{F}}} \\ &= \int d^{p+1}x e^{\frac{\kappa}{2\pi} \widehat{F}(x)}. \end{aligned} \tag{5.10}$$

This proves the identity (5.7) for $p = 3$. As a simple corollary, we also get

$$\begin{aligned} & \int d^{p+1}X \sqrt{\det(1 + F(X)\theta)} e^{\frac{\kappa}{2\pi} \mathbf{F}(X)} \\ = & \int d^{p+1}X e^{\frac{\kappa}{2\pi} F(X)} = \int d^{p+1}X \mathrm{ch}(E) \end{aligned} \tag{5.11}$$

In particular,

$$\int d^4x\sqrt{\det \mathrm{g}}\Big(\mathrm{g}^{-1}F\Big)\wedge\Big(\mathrm{g}^{-1}F\Big)=\int d^4xF\wedge F. \tag{5.12}$$

Noting that the instanton action is topological, i.e. independent of the background metric, the above identity together with Eq. (5.7) seems to perfectly agree with our interpretation on the induced metric.

The integrated Chern character can be expressed in terms of Chern-Simons form $\widehat{\Omega}$ such that

$$\int d^{p+1}x(\widehat{F}\wedge\cdots\wedge\widehat{F})_{(\mathrm{p+1})-\mathrm{form}}=\int d^{p+1}x\; d\widehat{\Omega}_p \tag{5.13}$$

with

$$\widehat{\Omega}_p=\frac{p-1}{2}\int_0^1 dt(\widehat{A}\wedge\widehat{F}_t\wedge\cdots\wedge\widehat{F}_t) \tag{5.14}$$

where $\widehat{F}_t=td\widehat{A}-it^2\widehat{A}\star\widehat{A}$. The proof of Eq. (5.13) is essentially the same as the ordinary non-Abelian case since the cyclic property is available under the integral. The exact SW map (5.3) together with the identity (5.10) implies that

$$\int d^px\;\widehat{\Omega}_p(\widehat{A},\widehat{F}_t)(x)=\int d^pX\;\Omega_p(A,F_t)(X). \tag{5.15}$$

If we definitely take care of the star products between $\widehat{F}$'s, the SW map (5.15) may not be quite true. For $p=3$, however, we have the following property

$$\int d^4x(\widehat{f}\star\widehat{g})(x)=\int d^4x(\widehat{f}\widehat{g})(x) \tag{5.16}$$

and, using the SW map (3.7) and the identity (5.12), we get

$$\begin{aligned}\int d^4x(\widehat{F}\wedge\widehat{F})(x) &= \int d^4X\sqrt{\det(1+F(X)\theta)}(\mathbf{F}\wedge\mathbf{F})(X)\\ &= \int d^4X(F\wedge F)(X).\end{aligned} \tag{5.17}$$

The same result also directly follows by taking the variation of the left side of Eq. (5.17) with respect to the NC parameter θ and showing that it vanishes [26]. Moreover the same map can be obtained by exploiting the SW map for anomalous axial currents [14]. Therefore the identity (5.15) is still true for $p=3$ as was proved earlier in [11]. It was also shown [27] that the equivalence persists at the quantum level in perturbation theory. For higher dimensions, e.g. $p=5$, NC ordering effect comes in, i.e.,

$$\int d^4x(\widehat{f}\star\widehat{g}\star\widehat{h})(x)\neq\int d^4x(\widehat{f}\widehat{g}\widehat{h})(x). \tag{5.18}$$

Thus we cannot simply replace $\widehat{F}(x)$ by $\mathbf{F}(X)$ due to the explicit derivatives of $\widehat{F}(x)$ in the star products. The NC ordering effect essentially spoils the property (5.15), the form invariance of the Chern-Simons action under the SW map, as was shown by Polychronakos [28].

6 Discussion

In this paper we revisited the dual description on DBI actions via the exact SW map recently obtained by one of us in [5]. We showed that the deformation quantization scheme clearly explains why the dual description via SW map includes a fluctuating geometry induced by gauge fields and noncommutativity, in a sense, reflects the presence of a fluctuating "medium". Furthermore the picture on the induced gravity has been generalized to an adjoint scalar field and has been particularly useful to understand topological invariants in NC field theories. This picture may have many interesting implications in both string theory and field theory.

Our discussions so far have been confined only to Abelian gauge group and to the DBI limit. Thus several interesting open issues remain for the future. First of all, it will be interesting to find non-Abelian generalization [29] and an exact SW map for DBI actions with derivative corrections [25, 30]. Let us discuss some issues briefly.

When one has N coincident type II Dp-branes, the worldvolume theory is a $U(N)$ gauge theory. The explicit construction of such an action is a difficult problem that is not yet completely settled. But one may simply adopt the non-Abelian DBI action proposed by Tseytlin [29] in terms of a symmetrized trace prescription over the Chan-Paton indices. Fortunately Terashima already showed [31] (see also [32]) the equivalence between the non-Abelian DBI action and its NC counterpart in the approximation of neglecting derivative terms, using the differential equation defining the SW map and the change of variables (2.12) and (2.13), following a similar recipe to [4]. In the rank one case, the exact SW map (2.17) in the DBI approximation has been obtained by simply applying the change of variables (2.12) and (2.13) to the commutative DBI action (2.10). Thus the exact SW map for the non-Abelian DBI action may also be obtained similarly. It should be straightforward to check this claim [33].

The exact SW map (2.19) raises several interesting issues. Apart from its implications for conformal anomalies already stated, it should be useful in analyzing renormalizability, UV/IR mixing, and unitarity of SW deformed electrodynamics. Also it would be an interesting question as to whether or not the equivalence implied by the SW map persists even in the nonperturbative level. This question is rather subtle. It was pointed out [4, 15] that the change of variables from $\widehat{A}_\mu$ to A_μ or vice versa has only a finite radius of convergence. This can be seen from Eq. (3.2) that when $\partial y^\alpha/\partial x^\mu$ has a zero eigenvalue, F must diverge. Similarly, when ω has a zero eigenvalue, $\partial y^\alpha/\partial x^\mu$ must diverge. Thus the SW map may not completely encode the topology of gauge fields. One example may be the level quantization of NC Chern-Simons theory for $U(1)$ gauge group [34, 28]. In Eq. (5.15), we showed the equivalence between the commutative and NC Chern-Simons theories. However, in the course of the derivation, it was implicitly assumed that both $\partial y^\alpha/\partial x^\mu$ and $\omega_{\mu\nu}$ are nonsingular, which is not completely true. Thus the nonperturbative aspects of the SW map remain as an interesting future problem.

NC soliton solutions have been studied via SW map in several contexts [35]. Along this

direction, it will be interesting to reexamine NC $U(1)$ instantons [36] in view of the SW deformed electrodynamics (2.19). As Eq. (5.17) implies the rigidity of the topological charge of instantons under the SW map, an instanton solution on NC space must ensure the existence of the corresponding commutative instanton. It may be explicitly checked by studying the self-duality equations for both sides of Eq. (2.19) which are related to each other by the SW map (3.7). The correspondence between the commutative and NC instantons seems to lead to an intriguing picture that NC $U(1)$ instantons may be identified with Abelian instantons on ALE space [37].

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